# A NOTE ON HILBERT'S OPERATOR

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The transformation

$$(1)\ \mathfrak{H}f = \frac{1}{\pi} P\,V \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \{f(x+t) - f(x-t)\}$$

is well known to have the following properties:

LEMMA 1.[1] *When $1<p<\infty$, then $\mathfrak{H}f$ is a continuous (bounded) linear transformation with both domain and range $L_p(-\infty, \infty)$, and $\mathfrak{H}^2 f = -f$.*

LEMMA 2.[2] *When $f(t) \in L_1(-\infty, \infty)$, then $\mathfrak{H}f$ exists for almost all $x$ in $(-\infty, \infty)$, but does not necessarily belong to $L_1(a, b)$, where $a, b$ are arbitrary numbers $(-\infty \leqq a < b \leqq \infty)$; however $(1+x^2)^{-1} |\mathfrak{H}f|^q \in L_1(-\infty, \infty)$ when $0<q<1$. When $f$ and $\mathfrak{H}f$ belong to $L_1(-\infty, \infty)$, then $\mathfrak{H}^2 f = -f$.*

The case $p=1$ appears to present the greatest difficulties. In the present note I shall deal with the set of elements $f(t) \in L_1(-\infty, \infty)$ for which $\mathfrak{H}f \in L_1(-\infty, \infty)$. In consequence of the lemmas, in this set or in $L_p(-\infty, \infty)$ $(1<p<\infty)$, $\mathfrak{H}f$ has no characteristic values other than $\pm i$. We shall start from the sets of characteristic functions and, incidentally, from the class $\mathfrak{H}_p$, the theory of which has been developed by E. Hille and J. D. Tamarkin; $\mathfrak{H}_p$ is the set of functions $F(z)$ $(z=x+iy)$ which, for $y>0$, are regular and satisfy the inequality

$$(2)\qquad \int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0<p<\infty$ or $p=\infty$, respectively, where $M$ depends on $F$ and $p$ only.[3] By $\mathfrak{K}_p$ we denote the corresponding class defined for $y<0$, and by $F(t)$, $G(t)$ the limit-functions[3] $(y\to 0;\ x=t)$ of elements $F(z) \in \mathfrak{H}_p$, $G(z) \in \mathfrak{K}_p$. By $\mathfrak{H}_p'$ and $\mathfrak{K}_p'$, respectively, we denote the two sets of those limit-functions, and by $\mathfrak{H}_p' \dotplus \mathfrak{K}_p'$ the smallest linear manifold

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[1] M. Riesz, Mathematische Zeitschrift, vol. 27 (1928), pp. 218–244.

[2] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937, §5.14. E. Hille and J. D. Tamarkin, Fundamenta Mathematicae, vol. 25 (1935), pp. 329–352. Comparing our notation with that of Hille-Tamarkin, we have $\mathfrak{H}f = -\tilde{f}$.

[3] Loc. cit., $1 \leqq p < \infty$. T. Kawata, Japanese Journal of Mathematics, vol. 13 (1936), pp. 421–430, $0<p<\infty$. The limit-functions exist for almost all $t$ in $(-\infty, \infty)$ and belong to $L_p(-\infty, \infty)$.