ON A FAMILY OF FOURIER TRANSFORMS

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A procedure leading to specific properties of the symmetric stable distribution functions results by subjecting certain generalizations of the Bessel functions $J_\alpha(t)$ to a limit process.[1] There arises the question as to the existence of the corresponding distributions in case the limit process involved is omitted. The object of this note is to delimit the conditions under which the answer is affirmative. It turns out that the situation is quite different from that resulting in the limiting case of stability.

Let $L(t; \phi)$ denote the Fourier transform,

$$L(t; \phi) = \int_{-\infty}^{\infty} e^{itx} d\phi(x), \qquad -\infty < t < \infty,$$

of a distribution function $\phi = \phi(x)$, $-\infty < x < \infty$, that is, of a monotone function satisfying the boundary conditions $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. If ϕ' denotes the derivative of ϕ (a derivative which necessarily exists and is finite almost everywhere), the Stieltjes integral $L(t; \phi)$ reduces to

$$L(t; \phi) = \int_{-\infty}^{\infty} e^{itx} \phi'(x) dx$$

if and only if ϕ is absolutely continuous; in which case $\phi'(x)$ is called the density of $\phi(x)$.

For a real or complex number z, let z_+ denote z or 0 according as z is or is not positive. Thus, if $\lambda > 0$ and $\mu > 0$, the even function $L = (1 - |t|^\lambda)_+^\mu$ of the real variable t represents a continuous curve in a (t, L)-plane, this symmetric curve being situated above or on the t-axis according as $|t| < 1$ or $1 \leqq |t| < \infty$. It should be noted for later reference that the curve has at $t = 0$ a cusp with a tangent perpendicular to the t-axis, a corner with two distinct finite slopes or a tangent parallel to the t-axis, according as $\lambda < 1$, $\lambda = 1$ or $\lambda > 1$, while μ is arbitrary.

If $\lambda = 1 = \mu$, the function $(1 - |t|^\lambda)_+^\mu$ is $1 - |t|$ or 0 according as $|t| < 1$ or $|t| \geqq 1$, and represents therefore the Fourier transform $L(t; \phi)$ of an absolutely continuous distribution function $\phi(x)$. This is

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[1] A. Wintner, *On a class of Fourier transforms*, American Journal of Mathematics, vol. 58 (1936), pp. 45–90.