THE ANALYSIS OF LINEAR TRANSFORMATIONS

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1. **Introduction.** If T is a closed additive transformation with domain dense (abbreviated c.a.d.d.) between Hilbert spaces, there exist two resolutions of the identity, $E_1(\lambda)$ and $E_2(\lambda)$ such that $E_2(\lambda)T \subset TE_1(\lambda)$. These resolutions are defined for $0 \leqq \lambda < \infty$, and if $0<\alpha<\beta$ T is an isomorphism, when contracted to the range[1] of $E_1(\beta)-E_1(\alpha)$. This is important in applications, since only under an isomorphism is the convergence of a sequence of elements equivalent to the convergence of the images. This property compensates for the lack of compactness of Hilbert space.

These resolutions also permit us to express T as a denumerable sum of such isomorphisms.[2] Each isomorphism in turn can be expressed in terms of the values of T on an orthonormal set complete in a certain subspace. Thus T is analysed into components which determine it by addition and closure. Another interesting property of these resolutions associated with T is the fact that if f is such that for every $g \in \mathfrak{M}_1(\lambda)$ (the range of $E_1(\lambda)$), $|f+g| \geqq |f|$ and if Tf exists then $|Tf+Tg| \geqq |Tf|$.

When one considers a c.a.d.d. T in a general reflexive Banach space,[3] many properties lose their significance, but those mentioned above do not. Since they indicate a complete analysis of such transformations one is led to consider the possibility of generalizing the notion of a resolution of the identity and the association of two of these with a c.a.d.d. T.

At least five such generalizations are possible. However, the complete analysis given above cannot be carried through in general linear spaces at present because of various unsolved problems of these spaces. In the present talk, we show the dependence of this analysis upon these problems and classify the problems from this point of view. It is hoped that this will result in a more systematic development of the theory of linear spaces.

2. **Projections.** The difficulties appear in attempting to generalize the notion of a projection. Let us consider the usual notion of a pro-

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[1] Cf. [11, chap. 9], or [8, §5, pp. 312–318, in particular Theorem VI, p. 315].

[2] Cf. [8, loc. cit.].

[3] We shall follow the notation of [1]. A reflexive space is one such that $\overline{\overline{\mathfrak{B}}} = \mathfrak{B}$.