ON APPROXIMATION BY EUCLIDEAN AND NON-EUCLIDEAN TRANSLATIONS OF AN ANALYTIC FUNCTION

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In 1929 G. D. Birkhoff established[1] the noteworthy result that an entire function $F(z)$ exists such that to an arbitrary entire function $g(z)$ corresponds a sequence $a_1, a_2, \cdots$ depending on $g(z)$ with the property

$$\lim_{n\to\infty} F(z + a_n) = g(z) \tag{1}$$

for all z, uniformly for z on every closed bounded set.

It is the object of the present note (a) to indicate that not merely an arbitrary entire function $g(z)$ can be expressed in the form (1), but also any function analytic in a simply connected region, and (b) to study the non-euclidean analogue of the entire problem; precisely analogous results are obtained. Some related topics under (a) have recently been studied by A. Roth,[2] who, however, does not mention the results to be proved here.

The immediate occasion of the interest of the present writers[3] in the problem is through (b), for non-euclidean translations have been widely used in the study of derivatives of univalent and other functions analytic in the unit circle $|z| = 1$; limit functions under such translations are of great significance in the study of derivatives and of limit values of a given function as a variable point z approaches the circumference $|z| = 1$.

We shall give a proof of the following theorem, proof and theorem differing only in detail from those of Birkhoff:

THEOREM 1. *There exists an entire function $F(z)$ such that given an arbitrary function $f(z)$ analytic in a simply connected region R of the z-plane, we have for suitably chosen $a_1, a_2, \cdots$ the relation*

$$\lim_{n\to\infty} F(z + a_n) = f(z) \tag{2}$$

for z in R, uniformly on any closed bounded set in R.

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[1] Comptes Rendus de l'Académie des Sciences, Paris, vol. 189, pp. 473–475.

[2] Comentarii Mathematici Helvetici, vol. 11 (1938–1939), pp. 77–125.

[3] Compare Seidel and Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, a forthcoming paper in the Transactions of this Society.