ON TRIGONOMETRICAL SERIES WHOSE COEFFICIENTS DO NOT TEND TO ZERO[1]

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Let $\sum \rho_n \cos(nx-\alpha_n)$ be a series (S) such that

$$\rho_n \geqq 0, \qquad \limsup \rho_n > 0. \tag{1}$$

It is well known that the set E of convergence of S is of measure zero, a result due to Cantor and Lebesgue. More recently Rajchman has proved that E is a sum of an enumerable sequence of H-sets, and so in particular (an H-set being closed and of measure zero) that E is of the first category.[2]

We propose to establish a more precise property of the sets E and to show the connection between these sets and sets of absolute convergence (that is, sets in which a trigonometrical series can converge absolutely without being absolutely convergent everywhere).

We propose to call, in memory of Rajchman, "set of the type R" any set E such that a series (S) exists which satisfies the condition (1) and converges in E. E being a sum of closed sets it is natural to investigate the properties of perfect sets of the type R. Let P be such a set. For every x belonging to P we must have $\lim \rho_n \cos(nx-\alpha_n)=0$. But by (1) there exists an infinite sequence of integers $\{n_k\}$ such that $\rho_{n_k}>r>0$. Hence,

$$\lim \cos(n_k x - \alpha_{n_k}) = 0. \tag{2}$$

We can assume, without loss of generality that the point $x=0$ belongs to P. Hence

$$\lim \cos \alpha_{n_k} = 0. \tag{3}$$

(2) and (3) give immediately

$$\lim |\sin n_k x \sin \alpha_{n_k}| = 0, \tag{4}$$

and as by (3)

$$\lim |\sin n_k x \cos \alpha_{n_k}| = 0 \tag{5}$$

we get, by adding (4) and (5),

$$\lim |\sin n_k x| = 0. \tag{6}$$

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[1] Presented to the Society, May 2, 1941.

[2] See Zygmund, *Trigonometrical Series*, Warsaw, 1935, pp. 267–270.