ON A THEOREM BY J. L. WALSH CONCERNING THE MODULI OF ROOTS OF ALGEBRAIC EQUATIONS

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In 1881 A. E. Pellet published[1] the following very useful theorem:

If the polynomial

$$F(z) \equiv |a_0| + |a_1|z + |a_2|z^2 + \cdots + |a_{k-1}|z^{k-1} - |a_k|z^k + |a_{k+1}|z^{k+1} + \cdots + |a_n|z^n, \tag{1}$$

$$0 < k < n,\ a_0a_n \neq 0,$$

has two positive roots x_1 and x_2 $(x_1 < x_2)$, then the polynomial

$$f(z) \equiv a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \tag{2}$$

has no roots in the annulus $x_1 < |z| < x_2$ and precisely k roots in the circle $|z| \leqq x_1$.

While Pellet's proof for his theorem utilizes the theorem of Rouché, J. L. Walsh published in 1924[2] another more direct proof and established in the same memoir a sort of converse of Pellet's theorem. To formulate his result, consider the set $\mathfrak{F}$ of all polynomials

$$f(z) \equiv a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \tag{3}$$

which correspond to given moduli of coefficients. All polynomials of $\mathfrak{F}$ are obtained from one of them, $f(z)$, if the factors $\epsilon_0, \epsilon_1, \cdots, \epsilon_n$ in the expression

$$\epsilon_0a_0 + \epsilon_1a_1z + \epsilon_2a_2z^2 + \cdots + \epsilon_na_nz^n \tag{4}$$

assume independently all values of modulus 1. Let $\mathfrak{M}$ be the set of roots of all polynomials in $\mathfrak{F}$. It is immediately seen that if $\mathfrak{M}$ contains a number α it also contains all numbers with the modulus $|\alpha|$.

It was proved by Cauchy that all roots of (4) lie on or within the circle $|z| = y_1$, where y_1 is the single positive root of the polynomial

$$|a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n$$

and that all roots of (4) lie on or are exterior to the circle $|z| = y_2$, where y_2 is the single positive root of the polynomial

[1] A. E. Pellet: *Sur un mode de séparation des racines des équations et la formule de Lagrange*, Bulletin des Sciences Mathématiques, (2), vol. 5 (1881), pp. 393–395.

[2] J. L. Walsh: *On Pellet's theorem concerning the roots of a polynomial*, Annals of Mathematics, vol. 26 (1924), pp. 59–64.