A CHARACTERIZATION OF THE GROUP OF HOMOGRAPHIC TRANSFORMATIONS

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1. **Introduction.** The objectives of this note are three-fold: (1) to present a new differential geometric characterization of the group of homographic transformations of a complex variable, (2) to interpret in geometrical language the significance of the invariance of the Schwarzian derivative under a homographic transformation, and (3) to characterize a general homographic transformation by its unique association with two families of concentric circles.

2. **Preliminaries.** Let the equation

$$w = w(z) \tag{1}$$

denote a conformal representation of the points $z = x + iy$ of a region R of the z-plane on the points $w = u + iv$ of a region $\overline{R}$ of the w-plane, whereby a general curve C is transformed into a curve $\overline{C}$. Let γ and $\bar{\gamma}$ denote the curvatures of C and $\overline{C}$ at corresponding points z and w, and let s and $\bar{s}$ denote corresponding lengths of arc of C and $\overline{C}$. For a given transformation (1) it is well known that the rate of variation $ds/d\bar{s}$ is a function $\lambda(x, y)$ which may be expressed in any one of the following forms $(u_x^2+u_y^2)^{-1/2}$, $(u_x^2+v_x^2)^{-1/2}$, $(u_y^2+v_y^2)^{-1/2}$, $(v_x^2+v_y^2)^{-1/2}$.

Comenetz[1] (using a different notation) has obtained, by elementary methods, the formula

$$\bar{\gamma} = \gamma\lambda + \lambda_y \cos\theta - \lambda_x \sin\theta, \tag{2}$$

wherein $\theta = \arctan(dy/dx)$, which is *the law of transformation of curvature in conformal mapping*, and the formula

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds + \lambda\left[\lambda_{xy}\cos 2\theta + \tfrac{1}{2}(\lambda_{yy} - \lambda_{xx})\sin 2\theta\right], \tag{3}$$

which is *the law of transformation of the rate of change of curvature with respect to arc length, under conformal mapping.*

3. **A new characterization of the group of homographic transformations.** It is known that the most general directly conformal transformation which carries circles into circles (including straight lines) is a homographic transformation. The transformation (1) will be such a transformation if and only if

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[1] Comenetz, *Kasner's invariant and trihornometry*, The American Mathematical Monthly, vol. 45 (1938), pp. 82–87.