NOTE ON INTERPOLATION

G. GRÜNWALD

Let

$$A_n\colon\ x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)}, \qquad -1 \leqq x_1^{(n)},\ 1 \geqq x_n^{(n)}, \tag{1}$$

denote a set of n distinct points of the interval $-1 \leqq x \leqq +1$. If $f(x)$ is a given function defined in $-1 \leqq x \leqq +1$ we call the polynomial

$$I_n(x; f) = \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) \tag{2}$$

an interpolation polynomial of $f(x)$ corresponding to the abscissas (1), where for the polynomials $q_1^{(n)}(x)$, $q_2^{(n)}(x)$, $\cdots$, $q_n^{(n)}(x)$

$$q_k^{(n)}(x_k^{(n)}) = 1, \qquad q_k^{(n)}(x_i^{(n)}) = 0, \qquad i \neq k. \tag{3}$$

Then the polynomial (2) represents a polynomial which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$. The polynomials $q_k^{(n)}(x)$ $(k = 1, 2, \cdots, n)$ are called the fundamental polynomials of the interpolation corresponding to the set A_n. We consider the sequence $I_n(x; f)$ $(n = 1, 2, \cdots)$ under the condition of continuity of $f(x)$.

Let $\omega_n(x)$ be a polynomial of degree n, not identically zero, vanishing at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$ and let

$$q_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \equiv l_k^{(n)}(x); \tag{4}$$

then (2) represents the nth Lagrange polynomial of $f(x)$ corresponding to the abscissas (1), which is the uniquely determined polynomial of degree $n-1$ which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$.

It is known that for a given arbitrary sequence $\{A_n\}$:

$$x_1^{(1)};\ x_1^{(2)},\ x_2^{(2)};\ x_1^{(3)},\ x_2^{(3)},\ x_3^{(3)};\ \cdots;\ x_1^{(n)},\ x_2^{(n)}, \cdots, x_n^{(n)};\ \cdots \tag{5}$$

there exists a continuous function $f(x)$ such that the sequence of the Lagrange interpolation polynomials is not uniformly convergent, even divergent at a preassigned point.[1]

[1] G. Faber, *Über die interpolarische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 389–408. S. Bernstein, *Sur la limitation des valeurs d'une polynome $P_n(x)$ de degré n sur tout un segment par ses valeurs en $(n+1)$ points du segment*, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050. In the important special case $x_k^{(n)} = \cos (2k-1)\pi/2n$, that is, for the zeros $\omega_n(x) = \cos n(\arc \cos x)$ ($\omega_n(x)$ is the nth Tschebycheff polynomial) much more is known. I have proved the existence of a continuous function $f(x)$ for which