METRIC SEPARABILITY AND OUTER INTEGRALS[1]

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R. L. Jeffery [1] investigated an upper integral for functions (from the line to real numbers) not necessarily measurable. Let f be bounded, $\alpha<f(x)<\beta$ for $x \varepsilon A$, let $e_i=E_x[a_i \leqq f(x)<a_{i+1}]$ where $\alpha=a_0<a_1< \cdots <a_n=\beta$ and consider $\sum a_i m^* e_i$. If as n increases and max $(a_{i+1}-a_i)$ approaches zero, the limit of this sum exists and is independent of the subdivisions, this limit is the upper integral of f over A, $\int_A^* f(x)dx$.

Two point sets with finite outer measure are metrically separable if the outer measure of their sum is the sum of their outer measures. A function is metrically separable on a set A if for each constant λ the part of A on which the function is greater than λ is metrically separable from the rest of A. Jeffery proved that metric separability may be made the basis of a comprehensive theory of integration which includes Young, Pierpont, and Lebesgue integration.

All measurable functions are metrically separable, but a function defined over a non-measurable set (and so necessarily non-measurable) may still be metrically separable. However, if f is metrically separable and possesses an outer integral on a set A, there exists a measurable set $B \supset A$ and a function ϕ, measurable on B and equal to f on A, such that $\int_B \phi = \int_A^* f$. If f is metrically separable and summable on A_1 and on A_2, it need not be metrically separable on A_1+A_2, but is summable on this set. Jeffery's methods of proving these results are not applicable, but the same results hold, as this paper shows, for functions from the plane to real numbers if Carathéodory outer linear measure and integration with respect to this measure are used.[2]

1. **Equivalence.** Let A be a set with finite outer linear (Carathéodory or Gillespie) measure and $\Gamma[A]$ the set of points of the complement of A where the superior outer density of A is positive. Then, [2], $\overline{A}=A+\Gamma[A]$ is a *massgleiche Hülle* of A, that is, is linearly measurable with linear measure equal to the outer linear measure of A. Thus A is linearly measurable if and only if $\Gamma[A]$ has linear measure zero, that is, $L^*\Gamma[A]=0$.

[1] Presented to the Society, in part December 30, 1936, under the title *Metric separability and the Hildebrandt integral*, and in part October 28, 1939, under the title *Metric separability*.

[2] The same methods also prove these results if outer Gillespie linear measure (see [3]) is used.