THE CONFORMAL NEAR-MOEBIUS TRANSFORMATIONS[1]

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1. Introduction. In a previous paper,[2] we discussed the point transformations of the plane with reference to the maximum number of circles preserved. A nonconformal point transformation of the complex plane converts at most $2\infty^2$ circles into circles.[3] A conformal transformation, not of the Moebius type, carries at most $2\infty^1$ circles into circles (excluding the $2\infty^1$ minimal lines which become minimal lines). A Moebius transformation carries the entire family of ∞^3 circles into circles. From these results, we obtain the following two characterizations of the group of Moebius transformations: (1) if $3\infty^2$ circles are carried into circles after a point transformation, then the same is true for all circles, and the point transformation is therefore a Moebius transformation; and (2) any conformal transformation which converts $3\infty^1$ circles into circles is a Moebius transformation.

In this paper, we shall determine the set of all *conformal near-Moebius transformations*. That is, we shall obtain the set of all conformal transformations which convert *exactly* $2\infty^1$ circles into circles. *Any conformal near-Moebius transformation is of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is any one of the three transformations* e^z, $\log z$, z^n. The two families preserved are two orthogonal pencils of circles.

The conformal near-collineation problem[4] is a special case of our problem. *Any conformal near-collineation is of the form* S_1TS_2 *where* S_1 *and* S_2 *are similitudes and* T *is any one of the three transformations* e^z, $\log z$, z^n. The family preserved is a pencil of straight lines (besides the $2\infty^1$ minimal lines).

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[1] Presented to the Society, February 24, 1940.

[2] Kasner and De Cicco, *Characterization of the Moebius group of circular transformations*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 209–213.

[3] In the previous paper, we derived these results for the point transformations of the *real* cartesian plane. But these same results may easily be derived for the complex cartesian plane without any difficulty. Note that a given family F of geometric configurations in the complex cartesian plane is said to possess ∞^n configurations if each of these is determined uniquely by n complex constants.

[4] Kasner, *The problem of partial geodesic representation*, Transactions of this Society, vol. 7 (1906), pp. 200–206. Also see Kasner, *The characterization of collineations*, this Bulletin, vol. 9 (1903), pp. 545–546; and Prenowitz, *The characterization of plane collineations in terms of homologous families of lines*, Transactions of this Society, vol. 38 (1935), pp. 564–599.