EXISTENCE AND DIFFERENTIABILITY THEOREMS FOR THE SOLUTIONS OF VARIATIONAL PROBLEMS FOR MULTIPLE INTEGRALS[1]

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I. INTRODUCTION

1. The object of the research [1] with which this address is concerned is twofold: first, to demonstrate, by direct methods, the existence of solutions, perhaps in some generalized sense, for a wide class of variational problems for multiple integrals, and second, to investigate further differentiability properties of the generalized solutions thus obtained. We consider only problems in nonparametric form.

2. Both of these goals have been achieved in the recent well known solutions of the problem of Plateau and in the solutions of the Dirichlet problem by variational methods. In more general problems this program has not been carried through completely except in very restricted cases. However, there are very important known results in connection with each separate aim. In connection with the existence theory, very important work has been done in the nonparametric case by Tonelli [2] and in the parametric case by McShane [3]. In the parametric case, practically nothing is known concerning the differentiability of the solutions obtained. In the nonparametric case, it has been proved by E. Hopf [4] that if $f(x, y, z, p, q)$ is of class C_α'' (that is, if f is of class C'' and its second derivatives satisfy a uniform Hölder condition[2] with exponent α, $0<\alpha<1$, on any bounded portion of space) and is the integrand in a regular variational problem (that is, if $f_{pp}f_{qq}-f_{pq}^2>0$, $f_{pp}>0$) and if z_0 is continuous on $\overline{G}$ and is of class C_β' on each region D with $\overline{D}\subset G$ and minimizes $\iint_G f(x, y, z, p, q)dxdy$ among all functions, continuous on $\overline{G}$, of class C' in G, and coinciding with z_0 on G^*, then z_0 is of class C_γ'' on any region D as above for any $\gamma<\alpha$. The author has proved in a previous paper [5] that the same result holds if z_0 merely satisfies a uniform Lipschitz condition[3] on regions $\overline{D}$ interior to G. The result of E. Hopf

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[2] A function $f(P)$ is said to satisfy a *uniform Hölder condition* on a set S if $|f(P_1)-f(P_2)|\leqq C\cdot|P_1P_2|^\alpha$, $0<\alpha<1$, for each pair of points (P_1, P_2) in S; C is called the *coefficient* and α is called the *exponent* of the Hölder condition and $|P_1P_2|$ denotes the distance from P_1 to P_2; C and α are supposed to be independent of P_1 and P_2.

[3] A function $f(P)$ is said to satisfy a uniform Lipschitz condition on a set S if $|f(P_1)-f(P_2)|\leqq C\cdot|P_1P_2|$ for every pair (P_1, P_2) on S, C independent of P_1, P_2.