NOTE ON COMPLEMENTED MODULAR LATTICES

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1. **Introduction.** In this note we study those elements of a complemented modular lattice whose complements are unique. We show that these elements are simply the neutral[1] elements of the lattice. It is also shown that an element with unique complement decomposes the lattice into a direct product of sublattices.[2] Hence if the lattice is indecomposable, each element not the null or unit element must have at least two complements. In case the lattice is of finite dimensions these results give a new proof of the Birkhoff-Menger[3] theorem that a complemented modular lattice of finite dimensions is a direct product of projective geometries and a Boolean algebra.

Although the existence of points and divisor-free elements is postulated, no chain conditions are assumed and the proofs are purely combinatorial.

2. **Notation and definitions.** Let $\mathfrak{S}$ denote a closed, complemented, modular lattice with null element z and unit element i. Complements of $a \varepsilon \mathfrak{S}$ will be denoted by $a', a'', \cdots$ and have the property that $(a, a')=i$, $[a, a']=z$. We assume that each $a\neq z$ divides a point p, and that each $b\neq i$ is divisible by a divisor-free element q. It follows that $a \supset b$, $a\neq b$, implies the existence of a point $p \subset [a, b']$ and of a divisor-free element $q \supset (b, a')$ such that $a \supset p$, $b \not\supset p$ and $q \supset b$, $q \not\supset a$. Hence each element of $\mathfrak{S}$ is the union of the points which it divides and the crosscut of its divisor-free divisors.

If S is a set of elements of $\mathfrak{S}$, $u(S)$ $(k(S))$ will denote the union (crosscut) of the elements of S. If $a \varepsilon \mathfrak{S}$ we denote the set of points p (divisor-free elements q) such that $a \supset p$ $(q \supset a)$ by P_a (Q_a). If $\mathfrak{S}$ is the direct product of the sublattices $\mathfrak{S}_1$ and $\mathfrak{S}_2$, we write $\mathfrak{S}=\mathfrak{S}_1\times\mathfrak{S}_2$.

An element a of $\mathfrak{S}$ is said to be *neutral* if $(a, [b, c]) = [(a, b), (a, c)]$ all $b, c \varepsilon \mathfrak{S}$. It is easily shown[4] that a is neutral if and only if $[a, (b, c)] = ([a, b], [a, c])$ all $b, c \varepsilon \mathfrak{S}$.

3. **Properties of elements with unique complements.** We need the following lemmas:

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[1] See §2.

[2] Added in proof: Theorems 3.1 and 4.1 are given by J. von Neumann in his *Continuous Geometrics* (Princeton).

[3] Garrett Birkhoff, Annals of Mathematics, (2), vol. 36 (1935), pp. 743–748; K. Menger, Annals of Mathematics, (2), vol. 37 (1936), pp. 456–481. Professor Birkhoff has informed the author that he has also obtained Theorem 4.2.

[4] O. Ore, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437.