ON THE UNIQUENESS OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS*

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It is well known that the existence of solutions of systems of differential equations can be established under hypotheses not strong enough to guarantee uniqueness of the solution. The standard device for ensuring uniqueness is to assume that the functions involved satisfy a certain Lipschitz condition. Graves† showed that, for systems consisting of a single equation, this could be replaced by a certain monotoneity requirement. In this note we shall establish a uniqueness theorem which contains both of these as special cases.

We suppose that $f^1(x, y), \cdots, f^n(x, y)$ are functions defined for all x in an interval $[a, b]$ and all points $(y^1, \cdots, y^n)$ of n-space; each $f^i(x, y)$ is assumed continuous in y for each fixed x, and measurable in x for each fixed y. Under these conditions it can be shown‡ that if there exists a function $S(x)$ summable over $a \leqq x \leqq b$ such that§

$$| f(x, y) | \leqq S(x),$$

then, for each x_0 in $[a, b]$ and each point y_0, there is an absolutely continuous function $y(x) \equiv (y^1(x), \cdots, y^n(x))$ such that $y(x_0) = y_0$, and‖

$$\dot{y}^i(x) = f^i(x, y(x)), \qquad a \leqq x \leqq b, \tag{1}$$

for almost all x. However, this solution may not be unique.¶

We therefore establish the following theorem:

THEOREM I. *Let the functions $f^i(x, y)$ be defined for all (x, y) with $a \leqq x \leqq b$. Let there exist a function $M(x)$ summable over $[a, b]$ such that for all x in $[a, b]$, all y and all η, the inequality*

$$\{f^i(x, y + \eta) - f^i(x, y)\}\eta^i \leqq M(x) | \eta |^2 \tag{2}$$

holds. Then, if $y_1(x)$ and $y_2(x)$ are absolutely continuous functions satisfying the differential equations (1) *for almost all x, and for some x_0 in*

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† L. M. Graves, *The existence of an extremum in problems of Mayer*, Transactions of this Society, vol. 39 (1936), pp. 456–471; in particular, p. 459.

‡ Carathéodory, *Vorlesungen über reelle Funktionen*, p. 672.

§ The symbol $|v|$ denotes the length of the vector v; thus $|f| = (f^if^i)^{1/2}$.

‖ The symbol $\dot{y}^i$ denotes the derivative $y^{i\prime}(x)$, where that derivative exists and is finite; elsewhere it has the value 0.

¶ Carathéodory, op. cit., p. 675.