ON SOME NEW CONGRUENCES IN THE THEORY OF BERNOULLI'S NUMBERS

N. G. W. H. BEEGER

For Bernoulli's numbers the following relations are known:

$$(h+1)^n = h^n; \qquad n > 1, \qquad h_1 = -\tfrac{1}{2}, \qquad B_n = (-1)^{n-1}h_{2n};$$
$$h_{2n+1} = 0 \quad \text{for} \quad n > 0.$$

For the symbol $k^n = h^{n+1}/(n+1)$ Kummer proved the congruence

$$k^a(1-k^b)^c \equiv 0 \pmod{(p^a, p^{ec})}, \tag{1}$$

p being a prime, $b = p^{e-1}(p-1)b_1$, $a+1 \not\equiv 0 \pmod{(p-1)}$. G. Frobenius* has given another proof of this congruence, without using infinite series. I shall now prove the congruence

$$(-1)^{i-1}k^{a+mb} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{m,s-1}C_{m-s,i-s}k^{a+(s-1)b} \pmod{p^i}, \qquad b = p-1, \tag{2}$$

which is equivalent to

$$(-1)^{i-1}\frac{B_{n+m\mu}}{2n+2m\mu} \equiv \sum_{s=1}^{i} (-1)^{s-1+(m-s+1)\mu} \cdot C_{m,s-1}C_{m-s,i-s}\frac{B_{n+(s-1)\mu}}{2n+2(s-1)\mu} \pmod{p^i}, \tag{3}$$

$C_{m,0} = 1$, $m \geqq i$, $i < 2n-1$, $2n \not\equiv 0 \pmod{(p-1)}$, $\mu = (p-1)/2$.

Take, in (1), $b = p-1$, $c = i$, $a = 2n-1$; then (1) gives

$$(-1)^{i-1}k^{a+bi} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{i,s-1}k^{a+(s-1)b} \pmod{p^i}.$$

Hence (2) is proved for the case $m = i$. Now suppose that (2) is proved for $m = i, i+1, i+2, \cdots, m$. From (1) it follows that

$$(-1)^m k^{a+(m+1)b} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{m+1,s-1}k^{a+(s-1)b} + \sum_{s=i+1}^{m+1} (-1)^{s-1}C_{m+1,s-1}k^{a+(s-1)b} \pmod{p^{m+1}}. \tag{4}$$

* Sitzungsberichte der Preussischen Akademie, vol. 39 (1910), p. 809