APOLARITY OF TRILINEAR FORMS AND PENCILS OF BILINEAR FORMS*

R. M. THRALL

1. **Introduction.** In this note the idea of apolar trilinear forms, introduced earlier by the author,† is generalized, and the main theorems are given new proofs independent of the group theoretic methods and notation used then. The importance of the apolarity concept in trilinear form classification is illustrated by a numerical application. In §3 there is given a new method of classifying singular pencils of bilinear forms based on Dickson's minimal numbers.‡

2. **Apolarity of trilinear forms.** A trilinear form $F(x, y, z) = \sum a_{hij}x_h y_i z_j$, where h runs from 1 to l, i from 1 to m, and j from 1 to n, has as *two-way rank invariants*§ r_h, r_i, r_j the smallest numbers of variables x, y, z, respectively, in terms of which the form can be expressed. Let F and $F' = \sum b_{hij}x_h y_i z_j$ be two trilinear forms in which the numbers of x's, y's, z's are r_h, m, n and r_h', m, n, respectively, where r_h and r_h' are the two-way h-rank invariants of F and F'.

DEFINITION. *F and F' are said to be h apolar (relative to m, n)|| if*

$$\sum_{i,j=1}^{m,n} a_{hij}b_{\nu ij} = 0, \qquad h = 1, \cdots, r_h;\ \nu = 1, \cdots, r_h', \tag{1}$$

and

$$r_h + r_h' = mn. \tag{2}$$

We define i and j apolarity analogously with respect to the i and j two-way rank invariants of F and F'. We shall apply the term *apolar* to two forms if they are h, i, or j apolar. The following theorems concerning h apolarity can, of course, be rephrased in terms of i and j apolarity.

We consider questions of existence and uniqueness properties of

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† *Metabelian groups and trilinear forms*, American Journal of Mathematics, vol. 60 (1938), pp. 383–415.

‡ *Singular case of pairs of bilinear, quadratic, or Hermitian forms*, Transactions of this Society, vol. 29 (1927), pp. 239–253.

§ Cf. R. Oldenburger, *On canonical binary trilinear forms*, this Bulletin, vol. 38 (1932), p. 385.

|| Evidently $m \geqq r_i$, $n \geqq r_j$. For most applications we take $m = r_i$, $n = r_j$.