ARITHMETICAL CONSEQUENCES OF A TRIGONOMETRIC IDENTITY

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1. *Identities.* The formulas for the product of s sines, t cosines, or s sines and t cosines, give immediate elementary proofs concerning representations of numbers in certain forms of degrees s, t, $s+t$. Here we consider s sines, where $s>1$. Let us write

$$\psi_n(x_1, \cdots, x_s) \equiv \sum e_2 \cdots e_s(x_1 + e_2x_2 + \cdots + e_sx_s)^n,$$

where $\sum$ refers to the 2^{s-1} possible sets $(e_2, \cdots, e_s)$, $e_j = \pm 1$, $(j=2, \cdots, s)$. Then, for $t>0$,

$$2^{2t-1}(-1)^t \prod_{j=1}^{2t} \sin x_j\theta = \cos \psi(x_1, \cdots, x_{2t})\theta,$$

$$2^{2t}(-1)^t \prod_{j=1}^{2t+1} \sin x_j\theta = \sin \psi(x_1, \cdots, x_{2t+1})\theta,$$

in which, after expansion of the (symbolic) cos, sin on the right, $\psi^n(\ )$ is to be replaced by $\psi_n(\ )$. Such expansions were discussed in detail in a previous paper.*

The results of equating coefficients of θ^{2t}, θ^{2t+1} in these can be combined at once into the single identity

$$(1) \qquad s!2^{s-1}x_1 \cdots x_s = \psi_s(x_1, \cdots, x_s).$$

For $s=3$, (1) is due to Gauss, for $s>1$, to Tardy, who proved a slightly different form of (1) otherwise.† In the same way we see that

$$(2) \qquad \psi_n(x_1, \cdots, x_s) = 0, \qquad n < s.$$

In (1) let $x_1=x$, $x_j=1$ $(j>1)$. If precisely p of $e_2, \cdots, e_s$ are $+1$, $e_2+ \cdots +e_s=2p+1-s$, $e_2 \cdots e_s=(-1)^{s-1-p}$. The p can be chosen in ${}_{s-1}C_p$ ways. Hence we have

$$(3) \quad (-1)^{s-1}s!2^{s-1}x = \sum_{p=0}^{s-1} (-1)^p\, {}_{s-1}C_p(x+2p+1-s)^s.$$

* Transactions of this Society, vol. 38(1926), pp. 129–148.

† Annali di Scienze Matematische e Fisiche, vol. 2(1851), pp. 287–291.