ON POINCARÉ'S RECURRENCE THEOREM

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1. *Introduction.* Let S be a space in which is defined a measure μ such that $\mu(S)=1$. Suppose we are given a one parameter group of one to one transformations T_t, $(-\infty<t<\infty)$, of S into itself, with the properties:

(1) $$T_sT_t = T_{t+s}.$$

(2) For any measurable set E and any t the set T_tE is measurable and $\mu(T_tE)=\mu(E)$.

The following extension of Poincaré's recurrence theorem was proved by Khintchine.*

For any measurable E and any $\lambda<1$,

$$\mu(E\cdot T_tE) \geqq \lambda(\mu(E))^2$$

for a set of values t that is relatively dense on the t axis.

In this paper we give an elementary proof of this statement.

2. *An Auxiliary Theorem.* We prove the following theorem from which the recurrence theorem is an immediate consequence and which is also interesting in itself.

Let S be a space with a measure μ such that $\mu(S)=1$ and let $E_1, E_2, \cdots$ be an infinite sequence of measurable sets in S, all having a measure not less than m. Then for any $\lambda<1$ there exist in the sequence two sets E_i and E_k such that

$$\mu(E_iE_k) \geqq \lambda m^2.$$

Let us suppose that $\mu(E_iE_k)<p$ for any i and k. If we put

$$F_1 = E_1,\quad F_2 = E_2 - E_2F_1,\quad F_3 = E_3 - E_3F_2 - E_3F_1,$$
$$\cdots,\quad F_n = E_n - E_nF_{n-1} - \cdots - E_nF_1,$$

no two of the sets F have common points and F_i is part of E_i. Therefore

$$\mu(F_1) \geqq m,\quad \mu(F_2) > m - p,\quad \mu(F_3) > m - 2p,$$
$$\cdots,\quad \mu(F_n) > m - (n-1)p,$$

* A. Khintchine, *Eine Verschärfung des Poincaréschen "Wiederkehrsatzes,"* Compositio Mathematica, vol. 1 (1934), pp. 177–179.