A THEOREM IN THE THEORY OF SUMMABILITY

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1. *Introduction.* Knopp* has developed a general integral form for linear methods of summability which includes as special cases all such methods which to date have found any considerable application. He establishes sufficient conditions for the regularity of his method but does not completely treat the question of their necessity. It may be of interest to observe that if Knopp's integral is interpreted in the sense of Lebesgue one may readily construct an example of his method which is regular in the class of bounded, measurable functions, but which does not satisfy condition (b)† of his regularity theorem. In conformity with Knopp's notation, let the curves $\mathfrak{C}_x$, $\mathfrak{C}_y$ be taken as the real axes $0 \leqq x < \infty$, $0 \leqq y < \infty$, respectively. Denote by $(\mathfrak{A})$ the class of all complex functions $f(x) \equiv f_1(x) + if_2(x)$ defined on $\mathfrak{C}_x$ such that $f_1(x)$, $f_2(x)$ are bounded and measurable on the interval $0 \leqq x \leqq X$ for every $X > 0$, and such that $\lim_{x\to\infty} f(x) \equiv L_f$ exists. Finally, let the function $K(x, y)$ be defined as $(-1)^j$ for $n-1 \leqq y < n$, $(n = 1, 2, 3, \cdots)$, $(j-1)/2^n \leqq x < j/2^n$, $(j = 1, 2, \cdots, 2^n)$; as $1/(y+1)$ for $1 \leqq x \leqq y+1$, $0 \leqq y < \infty$; and as zero for $0 \leqq y < x-1$, $1 < x < \infty$. Then for every $f(x) \subset (\mathfrak{A})$, $g(y) \equiv \int_0^{y+1} K(x, y) f(x)dx$ clearly exists on $\mathfrak{C}_y$, and we have

$$g(y) = \int_0^1 K(x, y) f(x)dx + \int_1^{y+1} \frac{1}{y+1} f(x)dx,$$

where the second integral tends to L_f as $y \to \infty$, since the function $1/(y+1)$ satisfies the conditions of Knopp's regularity theorem. Moreover, given $\epsilon > 0$, there exist step-functions $s_j(x)$ such that $\int_0^1 |f_j(x) - s_j(x)| dx < \epsilon$, $(j = 1, 2)$. Consequently, in view of $|K(x, y)| \leqq 1$, we have

* Knopp, *Zur Theorie der Limitierungsverfahren*, Mathematische Zeitschrift, vol. 31 (1929–30), pp. 97–127. To save space we assume that the reader is familiar with this paper.

† Loc. cit., p. 101.