ON THE INVARIANT CHARACTER OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS*

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1. *Equations Set Up*. We shall consider a differential equation

$$\operatorname{div} F = |F|, \tag{1}$$

where F is a vector and $|F|$ denotes its length. Its real character is seen better if we write u, v, w for the components of F and ρ for $|F|$; we have then the differential equation

$$u_x + v_y + w_z = \rho, \tag{2}$$

(subscripts throughout this discussion mean differentiation) with the condition

$$u^2 + v^2 + w^2 = \rho^2, \tag{3}$$

which shows that the relation imposed on F is quadratic.

2. *Auxiliary Quantities*. In an attempt to reduce the system to linear equations we introduce auxiliary quantities α, β, γ, δ (also functions of x, y, z), in terms of which we have

$$\begin{aligned} u &= 2(\alpha\delta + \beta\gamma), & w &= \alpha^2 + \beta^2 - \gamma^2 - \delta^2, \\ v &= 2(\alpha\gamma - \beta\delta), & \rho &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2. \end{aligned} \tag{4}$$

Condition (3) is satisfied identically by a substitution of these expressions into it, and it remains to determine the conditions imposed upon α, β, γ, δ by (2). Substitution gives

$$\begin{aligned} &\alpha\left(\delta_x + \gamma_y + \alpha_z - \frac{\alpha}{2}\right) + \beta\left(\gamma_x - \delta_y + \beta_z - \frac{\beta}{2}\right) \\ &+ \gamma\left(\beta_x + \alpha_y - \gamma_z - \frac{\gamma}{2}\right) + \delta\left(\alpha_x - \beta_y - \delta_z - \frac{\delta}{2}\right) = 0, \end{aligned} \tag{5}$$

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