LINEAR FUNCTIONAL OPERATIONS ON FUNCTIONS HAVING DISCONTINUITIES OF THE FIRST KIND*

BY H. S. KALTENBORN

The most general form of a linear continuous functional operation on a function $f(x)$ continuous on (a, b) has been expressed by F. Riesz† as a Stieltjes integral of $f(x)$ with respect to a function of bounded variation. By virtue of the extension theorem of Hahn,‡ this result can be extended to the case where $f(x)$ has discontinuities of the first kind on (a, b). In establishing this more general case, the modified form of the integral considered by B. Dushnik§ is used instead of the ordinary Stieltjes integral. The expression obtained for $T[f]$ reduces to that given by Riesz when the function $f(x)$ is continuous since in this case the modified integral agrees with the ordinary integral.

The following definitions and properties enter into the statement and proof of the theorem. Let f denote an element of a linear vector space S of norm $\|f\|$;¶ then an operation T on S is linear if

$$T[k_1f_1 + k_2f_2] = k_1T[f_1] + k_2T[f_2], \tag{1}$$

where the k's are any constants; and T is bounded if there exists an $M>0$ depending on T such that

$$| T[f] | \leqq M\|f\| \quad \text{for every } f \text{ of } S. \tag{2}$$

The condition of boundedness (2) is equivalent** to the property of continuity

* Presented to the Society, April 6, 1934.

† Annales de l'École Normale Supérieure, (3), vol. 31 (1914), pp. 9–14. Two new proofs of this theorem have been given recently by T. H. Hildebrandt and I. J. Schoenberg, Annals of Mathematics, vol. 34 (1933), pp. 317–319.

‡ See S. Banach, *Théorie des Opérations Linéaires*, 1932, pp. 55, 59.

§ Dissertation, University of Michigan, 1931.

¶ See T. H. Hildebrandt, *Linear functional transformations in general spaces*, this Bulletin, vol. 37 (1931), p. 186.

** See S. Banach, loc. cit., p. 54.