NOTE ON THE USE OF FRACTIONAL INTEGRATION OF BESSEL FUNCTIONS*

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1. *Introduction.* In a recent paper, W. O. Pennell† pointed out that under certain conditions fractional integration and differentiation of trigonometric functions will lead to Bessel functions and vice versa. Making use of this fact he was able to convert known expansions in sines and cosines into expansions in Bessel functions, and known expansions in Bessel functions into trigonometric expansions by the simple process of term-by-term differentiation and integration. In this note it is shown that in some cases fractional integration of Bessel functions leads to the squares of such functions. Use is then made of this fact in order to derive a number of expansions in squares of Bessel functions by fractionally integrating known expansions in Bessel functions.

2. *Fractional Integration of Bessel Functions.* Fractional integration‡ with the Heaviside operator p is defined by the equation

$$p^{-\nu}f(x) = \int_0^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} f(t)dt, \tag{1}$$

where $\nu > 0$. Fractional differentiation is given by

$$p^{\nu}f(x) = \frac{d^b}{dx^b}\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} f(t)dt, \tag{2}$$

where $\nu > 0$, $0 < c < 1$, b is a positive integer, and $\nu = b - c$.

If in the well known Neumann's integral§

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† W. O. Pennell, *The use of fractional integration and differentiation for obtaining certain expansions in terms of Bessel functions or of sines and cosines*, this Bulletin, vol. 38 (1932), pp. 115–122.

‡ For a bibliography on fractional integration and differentiation see H. T. Davis, *The application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142.

§ See G. N. Watson, *Theory of Bessel Functions*, p. 150.