ON THE MINIMIZING PROPERTY OF THE HARMONIC FUNCTION

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1. *Introduction.* Let D be a bounded connected open set and D^* its boundary and let $\bar{D}=D+D^*$. It is well known † that if the function $u(x, y)$ be harmonic on D, then $u(x, y)$ minimizes the Dirichlet integral

$$I[f] = \int\int_D [f_x^2 + f_y^2]dx\,dy$$

in the class of all functions $f(x, y)$ possessing piecewise continuous partial derivatives f_x and f_y and coinciding with $u(x, y)$ on the boundary D^*. But in certain recent discussions of the problem of Plateau ‡ essential use is made of a generalization of this theorem; it was necessary to know that the harmonic function $u(x, y)$ minimizes $I[f]$ in a larger class of functions than those with continuous derivatives. It has been suggested that a proof of this fact should be published; the present note carries out the suggestion. The method of proof is similar to that due to Lebesgue. As an application, a theorem is proved which is of some interest in the theory of curved surfaces.

The functions with which we shall be concerned are those which are, as we shall say, *absolutely continuous by sections* (abbreviated a.c.s.). A function $v(x, y)$, defined and continuous on a bounded open set D, will be said to be a.c.s. on D if it satisfies the following conditions:

(1 a) *for almost all values y_0 of y the function $v(x, y_0)$ is absolutely continuous on each interval of the line $y=y_0$ lying in D;*

(1 b) *for almost all values x_0 of x it is absolutely continuous on each interval of the line $x=x_0$ lying in D.*

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† H. Lebesgue, Société Mathématique de France, Comptes Rendus, (1913), p. 48. Hurwitz-Courant, *Funktionentheorie*, 2d ed., p. 424.

‡ E. J. McShane, *Parametrizations of saddle surfaces*, etc., Transactions of this Society, vol. 35 (1933), pp. 716–733. T. Radó, *The problem of Plateau*, vol. II, No. 3, of the Ergebnisse der Mathematik und ihre Grenzgebiete, p. 99.