# FUNCTIONS OF EXPONENTIAL TYPE*

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1. *Definition and First Properties.* If $f(x)$ is an analytic function which is regular at $x_0$ and $x_1$, then by expanding $f(x)$ in powers of $x-x_0$ and suitably dominating the expansion it is easily shown that

$$\limsup_{n=\infty} | f^{(n)}(x_0) |^{1/n} = \limsup_{n=\infty} | f^{(n)}(x_1) |^{1/n},$$

where the superscripts denote derivatives with respect to $x$. In the proof it is convenient to carry out the argument first for the case in which the first of these superior limits is finite. If these superior limits have the finite value $q$, $(q \geqq 0)$, then $f(x)$ is an integral function; in such a case we shall say that $f(x)$ is of exponential type† $q$. This terminology is justified by the following fundamental theorem.‡

THEOREM 1. *A necessary and sufficient condition that the integral function $f(x)$ shall be of exponential type $q$ is* (1) *that numbers $\sigma$ shall exist for which it is true that for every positive number $\epsilon$ there exists a quantity $M$, depending on $\epsilon$ and $\sigma$ in general but independent of $x$, such that for all (finite) values of $x$ we have*

$$|f(x)| < Me^{(\sigma+\epsilon)|x|}, \tag{1}$$

*and* (2) *that $q$ shall be the least possible value for such numbers $\sigma$. Moreover, when $f(x)$ is of exponential type $q$, we have*

$$| f^{(n)}(x) | < M(q + \epsilon)^n e^{(q+\epsilon)|x|}, \quad (n = 0, 1, 2, \cdots), \tag{2}$$

*where $M$ is independent of $x$ and $n$.*

The demonstration is readily constructed by aid of (2) which is easily proved by use of the expansion of $f^{(n)}(x)$ in powers of $x$.

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† The term *exponential type* is taken from a paper by G. Pólya, *Analytische Fortsetzung und konvexe Kurven*, Mathematische Annalen, vol. 89 (1923), pp. 179–191.

‡ See p. 361 of a paper by R. D. Carmichael, *Summation of functions of a complex variable*, Annals of Mathematics, vol. 34 (1933), pp. 349–378.