RECENT PROGRESS ON WARING'S THEOREM AND ITS GENERALIZATIONS*

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1. *Introduction and Summary.* The simplest theorem in question states that every positive integer is a sum of four integral squares. This is an example of a universal Waring theorem. The elaborate theory due to Hardy and Littlewood yields a number $C(s, n)$ beyond which every integer is a sum of s integral nth powers greater than or equal to zero. Since C is excessively large, their theory yields essentially only asymptotic theorems.

For several years the writer has been elaborating his idea that it is possible to supplement these asymptotic theorems and show that they hold also for all integers below C. The resulting new universal theorems are here first published.

The various aspects of Waring's problem may be compared with those of the theory of functions of a complex variable. Such a function may be studied in the neighborhood of infinity (corresponding to asymptotic Waring theorems), or in the neighborhood of the origin (compare our new results in §2), or over the whole plane (corresponding to universal Waring theorems). Analytic continuation of a function has its analog in our extension of a range for which s nth powers suffice to a larger range for which also s nth powers suffice. Such an extension is different from ascent to a still larger range for which $s+1$ nth powers suffice (§3).

2. *On the Ideal Limit for the Universal Waring Theorem.* It has been proved that every positive integer is a sum of nine cubes, and noted that 23 requires nine cubes (8, 8 and seven 1's), whence 9 is the ideal limit for cubes. For biquadrates (or fourth powers), 79, 159, 239, 319, 399, 479 and 559 require 19 biquadrates. The published table to 4100 shows that 19 biquadrates suffice to 4100. But the best theorem to date is that all integers are sums of 35 biquadrates. Hence it is only on evidence by tables that 19 is called the ideal limit for biquadrates.

By a decomposition (into nth powers $\geqq 0$), we mean a linear

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