NOTE ON TRIPLE SYSTEMS*

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A Steiner triple system is said to have the property E if (and only if) the presence of the triples (abu), (bcv), (ucx) implies the presence of the triple (avx) in the system. For such a triple system Th. Skolem† has proved in an elementary way that there exists such a system when and only when the number n of elements involved is of the form $2^{k+1}-1$, where k is a positive integer and that for every n of such form there exists but one triple system with the property E. H. Hasse,† by means of Abelian groups, has given a simple proof of this theorem of Skolem.

Since a Steiner triple system formed from the n elements $x_1, x_2, \cdots, x_n$ has the property that each pair x_ix_j of elements in the set belongs to one and just one triple of the set forming the triple system and since this system is now to be restricted so as to have the property E it follows that the elements $x_1, \cdots, x_n$ and the triples of the system may be taken to be the points and the lines respectively of a finite geometry in the sense of Veblen and Bussey,‡ as one may readily see by comparing the properties of the system with the postulates of the geometry. The property E of the system is equivalent to the transversal property in the geometry.

Since the geometry contains just three points on a line it follows from the general theorem of Veblen and Bussey concerning the existence of finite geometries that this geometry is precisely their $PG(k, 2)$ consisting of $2^{k+1}-1$ points arranged in lines of three points each. These triples of points, constituting the lines of the geometry, form the required triple system with the property E and the system is unique. Thus it follows that the theorem of Skolem is essentially a special case of the existence theorem for the finite geometries.

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† Th. Skolem, Norsk Matematisk Tidsskrift, vol. 13 (1931), pp. 41–51; H. Hasse, ibid., vol. 13 (1931), pp. 105–107.

‡ Veblen and Bussey, Transactions of this Society, vol. 7 (1906), pp. 241–259.