

# ZERO-FREE REGIONS OF LINEAR PARTIAL FRACTIONS\*

BY MORRIS MARDEN†

1. *Introduction.* The object of this paper is to determine simple regions in the plane which do not contain any zeros of the partial fraction

$$\Phi = \sum_{j=1}^n \frac{\alpha_j}{z - a_j},$$

when the  $\alpha_j$ 's are complex constants.

The case of real  $\alpha_j$ 's has already been adequately treated by Gauss, Lucas, Jensen, and Bôcher in connection with their study of the derivative of a polynomial and the jacobian of two binary forms. In terms of  $\Phi$ , their results may be stated as follows.

(a) If all the  $\alpha_j$ 's have the same sign, there are no zeros of  $\Phi$  outside of any convex polygon enclosing the points  $a_j$ . If all the  $a_j$ , in addition, lie on the line-segment  $AB$ , there are no zeros of  $\Phi$  except at points on  $AB$ ‡.

(b) If all the  $\alpha_j$ 's have the same sign, and the  $a_j$ 's are either real or in conjugate imaginary pairs, there are no imaginary zeros of  $\Phi$  at points outside of all the Jensen circles.¶ In any Jensen circle containing  $k$   $a_j$ 's, and not overlapping any other Jensen circle, there are at least  $k-1$  and at most  $k+1$  roots of  $\Phi$ .¶¶

\* Presented to the Society, October 27, 1928.

† National Research Fellow, at Princeton University.

‡ The result of which this theorem is an immediate consequence was first stated by Gauss, (Werke, vol. 3, p. 112), in 1816, but rediscovered by Lucas (Comptes Rendus, 1868). This theorem, with reference to the derivative of a polynomial, was first stated by Lucas, Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

¶ See §6 for definition. This part of the theorem was stated without proof by Jensen, Acta Mathematica, vol. 36 (1912), p. 190; proved by Walsh, Annals of Mathematics, vol. 22 (1920), pp. 128-144.

¶¶ Walsh, *ibid.*