POLYNOMIALS $f[\phi(x)]$ REDUCIBLE IN FIELDS IN WHICH $f(x)$ IS IRREDUCIBLE*

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1. *Introduction.* Professor Ritt recently had occasion to consider the irreducible polynomials which become reducible when each argument is replaced by a power of itself.†

His results suggest the related problem of determining all polynomials $\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m)$, such that $f[\phi_1, \cdots, \phi_m]$ is reducible, $f(x_1, \cdots, x_m)$ being irreducible. There is no such problem for functions of one variable, as every polynomial in a single variable can be factored into linear factors. If, however, we restrict ourselves to a field R, the problem arises: Given a polynomial $f(x)$ with coefficients in R and irreducible in R; to determine all polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is reducible in R. The present paper is devoted to a solution of this problem.

2. *Reducibility of $\phi(x)-x_i$ in R'.* Let

$$(1) \qquad f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

be a polynomial with coefficients in R and irreducible in R; and let $\phi(x)$ be an arbitrary polynomial with coefficients in R. An irreducible factor $A(x)$ of

$$(2) \qquad f[\phi(x)] = [\phi(x) - x_1] \cdots [\phi(x) - x_n]$$

has a root in common with one of the equations $\phi(x)=x_i$, say $\phi(x)=x_1$. Let $a_1(x)$ be the greatest common divisor of $A(x)$ and $\phi(x)-x_1$, and

$$(3) \qquad \begin{cases} \phi(x) - x_1 = a_1(x)b_1(x) \\ \quad A(x) = a_1(x)c_1(x), \end{cases}$$

* Presented to the Society, February 25, 1928.

† J. F. Ritt, *A factorization theory for functions* $\sum_{i=1}^{i=n} a_i e^{\alpha_i x}$, Transactions of this Society, vol. 29 (1927), pp. 584–596.