SQUARE-PARTITION CONGRUENCES*

BY E. T. BELL

1. *Introduction.* It is evident that the theory of partitions and that of the representation of an integer as a sum of squares must be closely interwoven since both originate in the elliptic theta and modular functions. In seeking the relations thus suggested, we find at the outset some remarkable types of congruences which deserve independent notice on account of their generality. Each congruence is to the odd prime modulus p; the most frequent type concerns the function expressing the number of ways in which an integer is a sum of p, $3p$, p^2, $3p^2$, p^s $(s > 0)$ or rp squares, where r is prime to p, and one of the following: the familiar denumerants of the classical theory of partitions; two new functions depending upon those partitions of an integer in which no part appears more than r times. Of the latter functions those corresponding to $r = 2, 3, 6$ play a central part in the entire theory. The subject is extensive. We shall give a sketch of the methods used sufficient for its systematic development. For the ϑ, q formulas see, e.g., Tannery-Molk, *Fonctions Elliptiques*, and note that we use Jacobi's theta notation (*Werke*, vol. 1, p. 501), so that π is omitted from ϑ_1'.

2. *Fundamental Identities.* In the usual notation $q_j = q_j(q)$,

$$(1)\qquad \begin{aligned} q_0 &= \Pi(1 - q^{2n}), & q_2 &= \Pi(1 + q^m), \\ q_1 &= \Pi(1 + q^{2n}), & q_3 &= \Pi(1 - q^m), \end{aligned}$$

extending to $n = 1, 2, 3, \cdots$, $m = 1, 3, 5, \cdots$, Euler's identities are

$$(2)\qquad q_1q_2q_3 = q_1(\sqrt{q})q_3 = 1, \qquad q_0 = \Sigma(-1)^n q^{3n^2+n},$$

Σ extending to $n = 0, \pm 1, \pm 2, \cdots$. Denote by $A_j(n, r)$ the coefficient † of q^{2n} in q_j^r $(j = 0, 1)$, of q^n in q_j^r $(j = 2, 3)$.

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† The properties of these coefficients have been discussed and a practicable method for their numerical computation given in a paper which will be published in the AMERICAN JOURNAL.