PROOF OF AN ARITHMETIC THEOREM DUE TO LIOUVILLE.

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1. The theorem, quoted from volume II, page 337, of Dickson's *History of the Theory of Numbers*, is as follows. *If $f(m)$, $F(m)$ are two arbitrary functions having definite values for $m = 1, 2, 3, \cdots$, and*

$$X_\mu(m) = \Sigma d^\mu f(d), \qquad Z_\mu(m) = \Sigma d^\mu F(d),$$

where each summation extends over all divisors d of m, then for any real or complex numbers μ, ν, and $\delta = m/d$, we have

$$(A) \qquad \Sigma d^{\mu-\nu} X_\nu(d) Z_\mu(\delta) = \Sigma d^{\mu-\nu} Z_\nu(d) X_\mu(\delta).$$

No reference to a proof being given, presumably none has been published. But Liouville remarks* that "there is an exceedingly simple method which I shall develop on another occasion, and which will lead us very rapidly by a kind of regular and general algorithm to the formula (A)." Without going into more detail than suffices for the proof of (A) we shall indicate the nature of an algorithm of this type. Arbitrary functions such as f, F which take definite values for integral arguments are called numerical functions All of the functions considered in this paper are of this kind.

2. Let (d, δ) denote any pair of conjugate divisors of m. Form the value of $\varphi_1(x)\psi(y)$, where φ_1 and ψ are arbitrary functions, for $(x, y) = (d, \delta)$, sum $\varphi_1(x)\psi(y)$ over all pairs (d, δ), and denote the result by $\Sigma_m \varphi_1(d)\psi(\delta)$. Let (δ_1, δ_2) denote any pair of conjugate divisors of δ, so that $m = d\delta$, $\delta = \delta_1\delta_2$, $m = d\delta_1\delta_2$, and put $\psi(m) = \Sigma_m \varphi_2(d)\varphi_3(\delta)$; whence

$$\Sigma_m \varphi_1(d)\psi(\delta) = \Sigma_m[\varphi_1(d)\Sigma_\delta \varphi_2(\delta_1)\varphi_3(\delta_2)] = \Sigma_m \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3)$$

the last summation extending over all triads (d_1, d_2, d_3) of divisors defined by $m = d_1d_2d_3$. If now (p, q, r), (i, j, k) are any permutations of $(1, 2, 3)$, it is obvious that

$$\Sigma_m[\varphi_p(d)\Sigma_\delta \varphi_q(\delta_1)\varphi_r(\delta_2)] = \Sigma_m \varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3),$$

which may be written in the purely symbolic form

$$\varphi_p \cdot \varphi_q\varphi_r = \varphi_i\varphi_j\varphi_k.$$

* Liouville, JOURNAL DES MATHÉMATIQUES, (2), vol. 3 (1858), p. 66.