A THEOREM ON THE VARIATION OF A FUNCTION.

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THE following is a well known theorem of differential geometry:

The differential quotient $d\phi/ds$ (ds is the element of arc) of a function $\phi(u, v)$ at a point on a surface varies in value with the direction from the point. It equals zero in the direction tangent to the curve $\phi = c$, and attains its greatest absolute value in the direction normal to this curve.*

This theorem admits of a generalization if we use a more comprehensive definition of length, a definition sometimes employed in the calculus of variations. Let then

$$S = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

be the generalized length of arc along a curve

$$(C) \qquad x = x(t), \quad y = y(t).$$

By reason of homogeneity conditions†

$$S = \int_{t_0}^{t_1} F(x, y, \cos\theta, \sin\theta)\sqrt{x'^2 + y'^2}dt$$
$$= \int_{s_0}^{s_1} F(x, y, \cos\theta, \sin\theta)ds,$$

in which

$$\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

Then

$$A = \left|\frac{d\phi}{dS}\right| = e\frac{\phi_x dx + \phi_y dy}{F(x, y, \cos\theta, \sin\theta)ds}$$
$$= e\frac{\phi_x \cos\theta + \phi_y \sin\theta}{F(x, y, \cos\theta, \sin\theta)},$$

* See Eisenhart, Differential Geometry, pp. 82–83.

† See Bolza, Vorlesungen über Variationsrechnung, p. 194.