AN ELEMENTARY DOUBLE INEQUALITY FOR THE ROOTS OF AN ALGEBRAIC EQUATION HAVING GREATEST ABSOLUTE VALUE.

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LET there be given an algebraic equation of the nth degree written

$$x^n + C_{n,1}a_1x^{n-1} + C_{n,2}a_2x^{n-2} + \cdots + a_n = 0,$$

where $C_{n,1}$, $C_{n,2}$, $\cdots$ denote binomial coefficients. Let x_1, x_2, $\cdots$, x_n denote its roots, and X the greatest absolute value of any of these roots.

From the equations which express the coefficients in terms of the roots

$$C_{n,k}a_k = (-1)^k\Sigma x_1x_2\cdots x_k \quad (k = 1, 2, \cdots n)$$

we infer at once that

$$|a_k| \leqq X^k.$$

In fact there are precisely $C_{n,k}$ terms on the right-hand side of the kth equation, each of which is not greater than X^k.

Hence, if α stands for the greatest of the quantities

$$|a_1|, \ |a_2|^{1/2}, \ \cdots, \ |a_n|^{1/n},$$

it is clear that we have $X \geqq \alpha$.

Also from the given algebraic equation we obtain directly

$$|x|^n \leqq C_{n,1}|a_1||x|^{n-1} + \cdots + |a_n|,$$

where x represents any of the quantities x_1, x_2, $\cdots$, x_n. Replacing $|x|$ by one of its possible values X, and each quantity $|a_k|$ by the quantity α^k, at least as great, we obtain

$$|X|^n \leqq C_{n,1}\alpha|X|^{n-1} + \cdots + \alpha^n,$$

which may be written

$$X^n \leqq (X + \alpha)^n - X^n.$$