SOME SPECIAL BOUNDARY PROBLEMS IN THE THEORY OF HARMONIC FUNCTIONS.

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1. IN his paper on "Le problème de Dirichlet dans une aire annulaire," *Rendiconti del Circolo Matematico di Palermo*, volume 33 (1912), pages 134–174, H. Villat has shown that when $u(\rho, \theta)$ is harmonic, uniform, and regular in the circular ring $R > \rho > r$, and subject to the boundary conditions

$$(1) \qquad u(R,\ \theta) = \Phi(\theta), \qquad u(r,\ \theta) = \Psi(\theta),$$

$\Phi(\theta)$ and $\Psi(\theta)$ being integrable for $0 \leqq \theta \leqq 2\pi$ and satisfying the condition

$$(2) \qquad \int_0^{2\pi} \Phi(\theta) d\theta = \int_0^{2\pi} \Psi(\theta) d\theta$$

necessary for the uniformity of $u(\rho, \theta)$, then this function is given by

$$(3) \qquad \begin{aligned} u(\rho, \theta) + iv(\rho, \theta) = \frac{i\omega}{\pi^2} \int_0^{2\pi} \Phi(\alpha) \frac{\sigma'}{\sigma}\left(\frac{\omega}{i\pi} \log \frac{\rho}{R} + \frac{\omega}{\pi}(\theta - \alpha) \right) d\alpha \\ - \frac{i\omega}{\pi^2} \int_0^{2\pi} \Psi(\alpha) \frac{\sigma_3'}{\sigma_3}\left(\frac{\omega}{i\pi} \mathrm{o\,g} \frac{\rho}{R} + \frac{\omega}{\pi}(\theta - \alpha) \right) d\alpha, \end{aligned}$$

where σ and σ_3 are the Weierstrass elliptic sigma functions, the periods ω and ω' satisfying the conditions

$$\omega \text{ and } \frac{\omega'}{i} \text{ real}, \quad \frac{\omega'}{i\omega} = \frac{1}{\pi} \log \frac{R}{r}.$$

Villat's proof consists in a somewhat lengthy discussion of the integrals in (3), similar to the ordinary treatment of Poisson's integral.

Fejér[*] has solved Dirichlet's problem for a circle without the use of Poisson's integral. In the present note, I propose

[*] L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.