THE GROUPS GENERATED BY THREE OPERATORS EACH OF WHICH IS THE PRODUCT OF THE OTHER TWO.

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LET s_1, s_2, s_3 represent any three operators of a finite group G which satisfy the three conditions

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_3s_1 = s_2.$$

These give rise to the following equations:

$$s_1s_2=s_2^{-1}s_1=s_2s_1^{-1}=s_3,\ s_2s_3=s_3s_2^{-1}=s_3^{-1}s_2=s_1,\ s_3s_1=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first continued equation it follows that s_1 and s_2 transform each other into their inverses and have a common square. From the second and third similar results follow with respect to s_2, s_3 and s_1, s_3 respectively. Hence s_1, s_2, s_3 are three operators such that each is transformed into its inverse by the other two. As any set of operators which fulfill the condition that each one is transformed into its inverse by all the others generates either the hamiltonian group of order 2^α or the abelian group of this order* and of type $(1, 1, 1, \cdots)$, it follows that s_1, s_2, s_3 generate one of the following four groups: identity, the group of order 2, the four-group, or the quaternion group. That is, if s_1, s_2, s_3 satisfy the three conditions imposed on them at the beginning of this paragraph, G must be one of these four groups, and it is evident that these operators may be so chosen that G is any one of these four groups.

If the given conditions are replaced by:

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_1s_3 = s_2,$$

there results the following system of continued equations:

$$s_1s_2=s_2^{-1}s_1=s_1^{-1}s_2=s_3,\ s_2s_3=s_3s_2^{-1}=s_2s_3^{-1}=s_1,\ s_1s_3=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first one of these it follows that s_2 is transformed into its inverse by s_1 and that the two operators s_1, $s_1^{-1}s_2$ are of order 2 since each of them is equal to its inverse. From the second and third it follows that s_2 is also transformed into its inverse

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