ON LAMÉ'S SIX EQUATIONS CONNECTED WITH TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

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LAMÉ* has shown for a triply orthogonal system of surfaces given by the parameters ρ, ρ_1, ρ_2 that if the square of the element of length is given by $ds^2 = H^2 d\rho + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$, where H, H_1, H_2 are certain functions of ρ, ρ_1, ρ_2, then H, H_1, H_2 must satisfy the following system of equations:

$$\frac{\partial^2 H}{\partial\rho_1\partial\rho_2} = \frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\frac{\partial H_1}{\partial\rho_2} + \frac{1}{H_2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_2}{\partial\rho_1} \tag{1}$$

and two others of the same type; (2), (3)

$$\frac{\partial}{\partial\rho_1}\left(\frac{1}{H_1}\frac{\partial H}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho}\left(\frac{1}{H}\frac{\partial H_1}{\partial\rho}\right) + \frac{1}{H_2^2}\frac{\partial H}{\partial\rho_2}\frac{\partial H_1}{\partial\rho_2} = 0 \tag{4}$$

with two others of this type. (5), (6)

Also if V is a function of x, y, z for which

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

he has shown that

$$\frac{\partial}{\partial\rho}\left(\frac{H_1H_2}{H}\frac{\partial V}{\partial\rho}\right) + \frac{\partial}{\partial\rho_1}\left(\frac{H_2H}{H_1}\frac{\partial V}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho_2}\left(\frac{HH_1}{H_2}\frac{\partial V}{\partial\rho_2}\right) = 0.$$

If the system of coordinates ρ, ρ_1, ρ_2 is isothermal, this equation must be satisfied by $V = \rho$, or by $V = \rho_1$, or by $V = \rho_2$. Hence $H_1H_2/H = Q^2$, where Q is a function of ρ_1 and ρ_2 only. Similarly $H_2H/H_1 = Q_1^2$, and $HH_1/H_2 = Q_2^2$, where Q_i is a function not involving the variable ρ_i. Hence $H = Q_1Q_2$, $H_1 = Q_2Q$, $H_2 = QQ_1$. The six equations given above transform into six in the variables Q. Lamé † gives a solution of

* Leçons sur les coordonées curvilignes (1859), pp. 76, 78.
† Loc. cit., p. 99.