ON THE DEFINITION OF LOGARITHMS.

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AFTER reading Professor Stringham's interesting paper* on "A Classification of Logarithmic Systems," and at his suggestion, I undertook to examine his definitions from the point of view of the theory of functions. The results of my investigation are embodied in the following paper. If I have taken the liberty of deriving some well-known formulæ, I hope that at least the point of view will prove interesting.

Starting from Riemann's definition of the logarithm as that function $\phi(z)$ which satisfies the equation

$$\phi(uz) = \phi(u) + \phi(z), \tag{1}$$

we can easily find an expression for the derivative of the logarithm in terms of the derivative of the independent variable as follows: † Differentiating equation (1) on the assumption that z is constant, we have

$$z\phi'(uz) = \phi'(u);$$

whence, if we write $u = 1$, it follows that

$$z\phi'(z) = \phi'(1).$$

The value of $\phi'(1)$ can be chosen at will and is characteristic for the system of logarithms under consideration. It is called the *modulus* of the system, and we shall denote it by $M = m \text{ cis } \beta$.

Let us write $\phi(z) = w$, and denote time-derivates by dots. The last equation obtained can then be written

$$\dot{w} = M\frac{\dot{z}}{z}. \tag{2}$$

Writing $z = r \text{ cis } \theta$, we find by differentiation

$$\dot{z} = \dot{r} \text{ cis } \theta + r(-\sin\theta + i\cos\theta)\dot{\theta} = (\dot{r} + ir\dot{\theta}) \text{ cis } \theta$$
$$= \left(\frac{\dot{r}}{r} + i\dot{\theta}\right)z, \tag{3}$$

whence follows immediately

$$\dot{w} = M\frac{\dot{z}}{z} = M\left(\frac{\dot{r}}{r} + i\dot{\theta}\right). \tag{4}$$

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* *American Journal of Mathematics*, vol. 14, p. 187.

† See, for example, Durège, "Theorie der Funktionen," 3. Aufl.