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Hilbert modular forms, by Eberhard Freitag. Springer-Verlag, Berlin and New York, 1990, 250 pp., $49.00. ISBN 3-540-50586-5

The Hilbert modular group is a straightforward generalization of the classical modular group $SL(2, \mathbf{Z})$ and is obtained by replacing the ring of rational integers by the ring of integers of a totally real algebraic number field. Let K be the field in question and O_K its ring of integers. The action of the classical modular group on the upper half-plane H in the complex numbers (via fractional linear transformations) generalizes to an action of the Hilbert modular group $SL(2, O_K)$ on H^n via the $n = [K : \mathbf{Q}]$ embeddings of the field K into the real numbers. Hilbert himself explained in his famous turn-of-the-century lecture at the International Congress in Paris why he was interested in this generalization [5]. His starting point was the well-known theorem of Kronecker(-Weber) which says that any number field which is abelian over $\mathbf{Q}$ is contained in a cyclotomic number field $\mathbf{Q}(\zeta_m)$ and the beautiful (though at that moment partly conjectural) extension of this to imaginary quadratic number fields by the theory of complex multiplication. Here Hilbert followed Kronecker who had expressed as his philosophy (his "Jugendtraum," cf. a letter to Dedekind of 1880 in *Werke*. V) that every abelian extension of an imaginary quadratic number field L is contained in a number field obtained by adjoining special values of elliptic and modular functions. Hilbert wanted to find the analytic functions which play the same role for arbitrary algebraic number fields as the exponential function does in the Kronecker-Weber theorem and the j-function does in the theory of complex multiplication. ("Ich halte dies Problem für eines der tiefgehendsten und weittragendsten Probleme der Zahlen und Funktionentheorie.") Hilbert aimed at nothing less than a theory of modular functions of several variables which should be as important in number theory and geometry as the theory of modular functions was at the end of the last century.

The group itself had appeared earlier in work of Humbert, when he investigated the moduli of abelian surfaces on which there exist extra curves (i.e., their Néron-Severi group $\neq \mathbf{Z}$). Also Picard had come across these groups.