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G-ELLIPTIC SYSTEMS AND THE GENUS ZERO PROBLEM FOR M_{24}

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1. Introduction

Toward the end of the 1970s J. H. Conway and S. P. Norton wrote a quite unique and influential paper [3] in which they produced incontrovertible evidence, based partly on ideas of J. McKay and J. G. Thompson [13], that the Fischer-Griess Monster simple group M was intimately related to the modular function $j(\tau)$ familiar from the theory of modular forms. Thus was so-called 'Moonshine' brought into the world. Among other things, it was conjectured that there is a natural infinite-dimensional complex vector space $V^\natural$ which carries a $\mathbf{Z}$-grading into finite-dimensional subspaces $V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$ such that each V_i is a $\mathbf{C}M$-module. Moreover the graded character $\mathrm{ch}_{V^\natural}(q) := \sum_{n\geq -1} \dim V_n q^n$ of $V^\natural$ should be the Fourier expansion of $j(\tau)$, namely $q^{-1} + 196884q + 21,493,760q^2 + \cdots$ if the indeterminate q is interpreted as $e^{2\pi i\tau}$ for τ in the upper half-plane H. There should in addition be an analogous interpretation for

$$\mathrm{tr}_{V^\natural}(g, q) := \sum_{n\geq -1} \mathrm{trace}_{V_n}(g)q^n$$

for each $g \in M$; namely, corresponding to g there is a discrete subgroup Γ_g of $SL_2(\mathbf{R})$ of genus zero i.e., the compactified orbit space $(\Gamma_g\backslash H)^*$ is a sphere, such that $\mathrm{tr}_{V^\natural}(g, q)$ is a modular function invariant under Γ_g which generates the field of all such functions over $\mathbf{C}$. (Briefly, $\mathrm{tr}_{V^\natural}(g, q)$ is a so-called 'hauptmodul'.)

This conjecture is known as the genus zero problem for M and has recently been settled in the affirmative. The work of Frenkel-

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