BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 24, Number 2, April 1991

SOME APPLICATIONS OF GELFAND PAIRS TO NUMBER THEORY

BENEDICT H. GROSS

The classical theory of Gelfand pairs has found a wide range of applications, ranging from harmonic analysis on Riemannian symmetric spaces to coding theory. Here we discuss a generalization of this theory, due to Gelfand-Kazhdan, and Bernstein, which was developed to study the representation theory of p-adic groups. We also present some recent number-theoretic results, on local ϵ-factors and on the central critical values of automorphic L-functions, which fit nicely into this framework.

1. Compact pairs

Let G be a compact topological group. By a representation of G we will mean a continuous homomorphism from G to the group of unitary operators on a Hilbert space V. If V and W are two representations of G, the complex vector space $\mathrm{Hom}_G(V, W)$ consists of all continuous linear transformations from V to W which commute with the action of G.

We say V is an irreducible representation of G if and only if there are no nontrivial closed subspaces of V which are G-invariant. The irreducible representations of G are all finite dimensional [D, Chapter 3]. Let V be a fixed irreducible representation. Then V has, up to scaling, a unique G-invariant Hermitian structure, and any linear map from V to a Hilbert space W is continuous. If W is a representation of G, we define the multiplicity of V in W as the dimension of the complex vector space $\mathrm{Hom}_G(V, W)$. We will only consider those representations W (often called admissible) such that $d_i = \dim \mathrm{Hom}_G(V_i, W)$ is finite, for all irreducible representations V_i of G. In this case, W decomposes as a Hilbert space direct sum: $W \simeq \hat{\bigoplus}_i d_i V_i$. We say W is multiplicity-free if $d_i \leq 1$ for all i.

Received by the editors July 5, 1990 and, in revised form, September 13, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20C15, 11F66, 11F67.