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# SHAPE OPTIMIZATION FOR DIRICHLET PROBLEMS: RELAXED SOLUTIONS AND OPTIMALITY CONDITIONS

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ABSTRACT. We study a problem of shape optimal design for an elliptic equation with Dirichlet boundary condition. We introduce a relaxed formulation of the problem which always admits a solution, and we find necessary conditions for optimality both for the relaxed and the original problem.

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n (n \geq 2)$, let $f \in L^2(\Omega)$, and let $g: \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function (i.e. $g(x, s)$ measurable in $x$ and continuous in $s$) such that

$$|g(x, s)| \leq a_0(x) + b_0|s|^2 \qquad \forall (x, s) \in \Omega \times \mathbf{R},$$

for suitable $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbf{R}$. We consider the following optimal design problem:

$$\min_{A \in \mathscr{A}(\Omega)} \int_\Omega g(x, u_A(x))\, dx, \tag{1}$$

where $\mathscr{A}(\Omega)$ is the family of all open subsets of $\Omega$, and $u_A$ is the solution of the Dirichlet problem

$$-\Delta u_A = f \text{ in } A, \qquad u_A \in H_0^1(A), \tag{2}$$

extended by 0 in $\Omega \backslash A$.

It is well known that, in general, the minimum problem (1) has no solution (see for instance Example 2). The reason is that, although the solutions $u_{A_h}$ of (2) corresponding to a minimizing sequence $(A_h)$ of (1) always admit a limit point $u$ in the weak (not necessarily in the strong) topology of $H_0^1(\Omega)$, we can not find, in general, an open subset $A$ of $\Omega$ such that $u = u_A$. On the contrary, it can be proved (see [4]) that the limit function $u$ is the solution of a relaxed Dirichlet problem of the form

$$-\Delta u + \mu u = f \text{ in } \Omega, \qquad u \in H_0^1(\Omega) \cap L^2(\Omega; \mu), \tag{3}$$

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