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# HURWITZ GROUPS: A BRIEF SURVEY

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ABSTRACT. Hurwitz groups are the nontrivial finite quotients of the $(2, 3, 7)$ triangle group $\langle x, y | x^2 = y^3 = (xy)^7 = 1\rangle$. This paper gives a brief survey of such groups, their significance, and some of their properties, together with a description of all examples known to the author.

## 1. INTRODUCTION

A *Hurwitz* group is any finite group which can be generated by an element $x$ of order 2 and an element $y$ of order 3 whose product $xy$ has order 7. Equivalently, a Hurwitz group is any finite nontrivial quotient of the $(2, 3, 7)$ triangle group, that is, the infinite abstract group $\Delta$ with presentation $\Delta = \langle x, y | x^2 = y^3 = (xy)^7 = 1\rangle$. The significance of the latter group (and its quotients) is perhaps best explained by referring to some aspects of the theory of Fuchsian groups, hyperbolic geometry, Riemann surfaces, and triangle groups, as given below. Details may be found in the recent books by Beardon [1] and Jones and Singerman [15].

A *Fuchsian* group is any discrete subgroup of $\mathrm{PSL}(2, \mathbf{R})$, the group of all linear fractional transformations of the form $z \mapsto (az+b)/(cz+d)$ with $a, b, c, d \in \mathbf{R}$ satisfying $ad-bc = 1$. The latter group acts on the upper half-plane $\mathscr{U} = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$, in fact as the group of all conformal homeomorphisms of $\mathscr{U}$, and when $\mathscr{U}$ is endowed with the hyperbolic metric (given by $ds^2 = (dx^2+dy^2)/y^2$ for $z = x+iy \in \mathbf{C}$), $\mathscr{U}$ becomes a model of the hyperbolic plane, and $\mathrm{PSL}(2, \mathbf{R})$ acts as a group of hyperbolic isometries.

Any given Fuchsian group $\Gamma$ acts properly discontinuously on $\mathscr{U}$, and the quotient space $S = \mathscr{U}/\Gamma$ is a Riemann surface. Conversely, every Riemann surface can be obtained in this way. A *fundamental region* for $\Gamma$ is a closed set $F$ in $\mathscr{U}$ whose $\Gamma$-images

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