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# THE ADAPTION PROBLEM FOR APPROXIMATING LINEAR OPERATORS

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In this note we answer two open questions on the finite-dimensional approximation of linear operators in Banach spaces. The first result establishes bounds on the ratio $\alpha$ of the error of adaptive approximations to the error of nonadaptive approximations of linear operators (see [PW], open problem 1); terms are defined more precisely below. This result is of interest partly because of its connection to questions regarding the error arising in parallel computational solutions of linear problems in infinite dimension (see [TWW], [PW]).

The second result concerns (possibly nonlinear) continuous finite-dimensional approximations of infinite-dimensional linear operators in Banach space (see [KW]). It is shown that such approximations can yield strictly smaller error than even optimal linear ones. This statement has been shown to be false in Hilbert space (cf. [KW]).

We defer discussion of related results to give some precise definitions. Let $S: F \to G$ be a bounded linear operator from a linear space $F$ to a Banach space $G$. We wish to evaluate $S$ at an element $f \in F$ (the "problem element"), restricted to lie in a bounded balanced convex subset $B$ of $F$. The element $f$ is uncertain to the extent that it is specified only by the value of its image $N(f)$ under a finite rank operator $N$ (to be defined below). We induce a norm on $F$ whose unit ball is $B$.

Let $N: F \to Y$ be a linear operator (*information operator*), with $Y \equiv \mathbf{R}^n$ finite dimensional. Decompose $N$ into component linear functionals, $N = (l_1, l_2, l_3, \dots, l_n)$. The image $Nf$ represents the (finite-dimensional) information available about the (high or infinite-dimensional) problem element $f$.

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