BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 2, April 1990

0273-0979/90 $1.00 + $.25 per page

Determinantal rings, by Winfried Bruns and Udo Vetter. Springer Lecture Notes 1327, Springer-Verlag, Berlin, Heidelberg, New York, 1988, vii+236 pp., $20.00. ISBN 3-540-19468-1

Determinantal varieties were considered for the first time in the nineteenth century in connection with the first and second fundamental theorem of invariant theory.

Let us consider a vector space F of dimension r over a field k. Let Z denote the space of $(m+n)$-tuples

$$(x, \xi) = (x_1, \dots, x_m, \xi_1, \dots, \xi_n)$$

of m vectors and n covectors (i.e., $x_i \in F$, $\xi_j \in F^*$). We consider the natural action of the group $\mathrm{GL}(r)$ of linear automorphisms of F on Z. Then the first fundamental theorem of invariant theory says that the ring of $\mathrm{GL}(r)$-invariant polynomial functions on X is generated by natural invariants $X_{ij}(x, \xi) = \xi_j(x_i)$ for $1 \le i \le m$, $1 \le j \le n$. The second fundamental theorem describes the ideal of relations between X_{ij}'s. It is generated by the $r+1$-order minors of the $m \times n$ matrix $X = (X_{ij})$.

It turned out later that the ideals generated by minors of a matrix appear naturally in algebraic geometry. For example, the singular locus of a variety is "naturally" defined by the vanishing of minors of the proper order of the Jacobian matrix.

Ideals of this type were studied systematically for the last thirty years from an algebraic point of view. Let B be a commutative ring, $X = (X_{ij})$ be the $m \times n$ matrix of indeterminates over B. Then the determinantal ring is the factor ring $R_t(X) = B[X]/I_t(X)$, where $B[X]$ is the ring of polynomials in indeterminates X_{ij} with coefficients in B, and $I_t(X)$ is the ideal in $B[X]$ generated by the minors of order t of the matrix X.

The investigation of such rings became one of the central topics in commutative algebra. This research retained its importance for invariant theory since the action of $\mathrm{GL}(r)$ described above is one of very few classical actions for which the ring of invariants can be described explicitly. It became a rule that theorems proved about rings of invariants were first proved for determinantal varieties.