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A SHARP COUNTEREXAMPLE ON THE REGULARITY OF Φ-MINIMIZING HYPERSURFACES

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A standard problem in the calculus of variations seeks a hypersurface S of least area bounded by a given $(n-2)$-dimensional compact submanifold of $\mathbf{R}^n$. More generally, given any smooth norm Φ on $\mathbf{R}^n$, seek to minimize

$$\Phi(S) = \int_S \Phi(\mathbf{n}),$$

where $\mathbf{n}$ is the unit normal vector to S. Think of the integrand Φ as assigning a cost or energy to each direction. We assume that Φ is *elliptic* (*uniformly* convex), the standard hypothesis for regularity.

Geometric measure theory (cf. [M, Chapters 5, 8], [F 1, 5.1.6, 5.4.15]) guarantees the existence of a (possibly singular) Φ-minimizing hypersurface with given boundary. For the case of area ($\Phi(\mathbf{n}) = 1$), area-minimizing hypersurfaces are regular embedded manifolds up through $\mathbf{R}^7$, but sometimes have singularities in $\mathbf{R}^8$ and above. For general elliptic Φ, a result of Almgren, Schoen, and Simon [Alm S S, Theorem II.7] guarantees regularity up through $\mathbf{R}^3$, but there were no examples of singularities below $\mathbf{R}^8$. We establish the sharpness of the Almgren–Schoen–Simon regularity result by giving a singular Φ-minimizing hypersurface in $\mathbf{R}^4$.

The surface is the cone C over the Clifford torus $S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$:

$$C = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| = |y| \le 1\}.$$

The norm Φ depends smoothly on $\theta = \tan^{-1}(|y|/|x|)$ alone, so that we may view Φ as a norm on $\mathbf{R}^2$. The unit Φ-ball is pictured in Figure 1. Any smooth, symmetric, uniformly convex approximation of the square will do. Note that Φ is smaller (say 1) on

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