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Harmonic analysis on symmetric spaces and applications, by Audrey Terras. Springer-Verlag, Berlin, Heidelberg and New York, 1985, xii + 341 pp., $39.00 (paper), vol. 1. ISBN 0-387-96159-3 and 1988, x + 385 pp., $45.00 (paper), vol. 2 ISBN 0-387-96663-3

This two-volume set is a very informal, relatively elementary, and occasionally entertaining survey of parts of a highly-developed and very useful part of contemporary mathematics. Differential equations, special functions, number theory, physics, and statistics all make essential use of harmonic analysis (interpreted sufficiently broadly).

A prototype for a "symmetric space" is the unit circle $\mathbf{R}/\mathbf{Z}$; harmonic analysis is Fourier analysis. The fundamental idea is that *periodic functions can be "represented by" Fourier series.* Several things can be said about a Fourier series representation

$$f(x) \sim \sum_{n\in\mathbf{Z}} c_n e^{2\pi inx}.$$

For f square integrable this is an equality in an L^2-sense, and

$$c_n = \langle f, \psi_n\rangle \qquad \langle f, f\rangle = \sum_n |c_n|^2 \qquad \left(\psi_n(x) = e^{2\pi inx}\right).$$

Further, for another square-integrable function φ with Fourier series

$$\varphi(x) \sim \sum_{n\in\mathbf{Z}} d_n e^{2\pi inx}$$

we have the *Parseval identity*

$$\langle f, \varphi\rangle = \sum_n c_n \overline{d}_n.$$

Pointwise convergence is more delicate; the series converges to f at points where f satisfies a Lipschitz condition, and convergence is absolute and uniform if f is smooth. If f is smooth, then the Fourier coefficients of f are rapidly decreasing.

Some things are so simple in this paradigmatic example that they may not be noticeable. First, the functions $x \to e^{2\pi inx}$ (with $n \in \mathbf{Z}$) are the eigenfunctions for the one-dimensional Laplace operator $\Delta: f \to \partial^2 f/\partial x^2$. Integration by parts yields the self-adjointness of Δ; therefore, we might attempt to express $L^2(\mathbf{R}/\mathbf{Z})$