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Number theory has its foundation in the Fundamental Theorem of Arithmetic which states that every integer $n > 1$ can be written uniquely in the form

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \qquad p_1 < p_2 < \cdots < p_r, \tag{1}$$

where the p_i's are primes and the α_i's positive integers. From an algebraic point of view, this result describes completely the set of positive integers as a free semigroup generated by the primes. However, for many problems in analytic number theory one would like to have more information about the structure of the prime factorization (1) and, in particular, the number and the size of the prime factors involved.

The prime factorization of an integer can of course take quite different shapes. On the one hand, if n itself is a prime, then its prime factorization consists of a single prime raised to the first power. On the other hand, if n is of the form $k!$, say, then it has a prime factorization with many small primes and relatively large exponents α_i. However, these are extreme cases that apply only to a relatively sparse set of integers n, and one might ask if it is possible to describe more precisely the prime factorization of a "typical," or "random," integer. This turns out to be the case; in fact, the study of such questions has led to the development of a new branch of number theory, called probablilistic number theory.

The first result in this direction, obtained in 1917 by Hardy and Ramanujan [HR], showed that a "random" integer n has about $\log\log n$ prime factors in the following sense: Let $\omega(n)$ denote the number of distinct prime factors of n, so that $\omega(n) = r$ in the representation (1). Let $\psi(n)$ be a function of n tending to infinity arbitrarily slowly, as $n \to \infty$. Then the inequality

$$|\omega(n) - \log\log(n)| \leq \psi(n)\sqrt{\log\log n} \tag{2}$$

holds for "almost all" positive integers n in the sense that the proportion of positive integers $n \leq N$ for which (2) holds tends to one, as $N \to \infty$.