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A GENERALIZATION OF SELBERG'S BETA INTEGRAL

ROBERT A. GUSTAFSON

ABSTRACT. We evaluate several infinite families of multidimensional integrals which are generalizations or analogs of Euler's classical beta integral. We first evaluate a q-analog of Selberg's beta integral. This integral is then used to prove the Macdonald-Morris conjectures for the affine root systems of types $S(C_l)$ and $S(C_l)^\vee$ and to give a new proof of these conjectures for $S(BC_l)$, $S(B_l)$, $S(B_l)^\vee$ and $S(D_l)$.

1. INTRODUCTION

In 1944, A. Selberg [23] evaluated the following integral (see also Aomoto [1]):

$$(1)\quad \int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le n} |t_i - t_j|^{2z} \prod_{j=1}^n t_j^{x-1}(1-t_j)^{y-1} dt_j = \prod_{j=1}^n \frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)},$$

where n is a positive integer, $x, y, z \in \mathbf{C}$ and $\mathrm{Re}(x)$, $\mathrm{Re}(y) > 0$ and $\mathrm{Re}(z) > -\max\{\frac{1}{n}, \mathrm{Re}(x)/(n-1), \mathrm{Re}(y)/(n-1)\}$. For $n = 1$, the integral (1) reduces to Euler's classical beta integral.

Now let $n \ge 1$ and $a_1, a_2, a_3, a_4, b, q \in \mathbf{C}$ with

$$\max\{|a_1|, \ldots, |a_4|, |b|, |q|\} < 1.$$

For $c \in \mathbf{C}$ define

$$[c; q]_\infty = [c]_\infty = \prod_{k=0}^\infty (1 - cq^k).$$

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