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PRODUCT FORMULAS, HYPERGROUPS, AND THE JACOBI POLYNOMIALS

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If $\mathscr{P} = \{p_n\}_{n\in\mathbf{N}_0}$ ($\mathbf{N}_0 = \{0, 1, 2, \ldots\}$) is a sequence of orthogonal functions on a real interval I, we say that $\mathscr{P}$ has a *product formula* if for each s, t in I, there is a Borel measure $\mu_{s,t}$ with $\operatorname{supp}(\mu_{s,t}) \subseteq I$ such that

$$(1) \qquad \int_I p_n\, d\mu_{s,t} = p_n(s)p_n(t)$$

for every n in $\mathbf{N}_0$. Such formulas are important because they give rise to a variety of measure algebras and the means to study their harmonic analysis. An important class of such formulas was established by Gasper [8] for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which are orthogonal on $[-1, 1]$ with respect to the weight $(1-x)^\alpha \times (1+x)^\beta\, dx$. These include Chebyshev, Legendre, and ultraspherical or Gegenbauer polynomials as special cases. The product formula for Jacobi polynomials has 1 as an *identity* in the sense that for all t in $[-1, 1]\mu_{1,t}$ is the unit point mass concentrated at t, and it has *continuous support* in the sense that $\operatorname{supp}(\mu_{s,t})$ is a continuous function of (s, t). Moreover, the measures $\mu_{s,t}$ are all positive if and only if

$$(2) \qquad \alpha \geq \beta > -1 \text{ and either } \beta \geq -1/2 \text{ or } \alpha+\beta \geq 0.$$

It is natural to ask which orthogonal polynomials have such product formulas. The answer is a converse to Gasper's result:

Theorem 1. *If a family $\mathscr{P}$ of orthogonal polynomials has a product formula with identity, continuous support, and nonnegative measures $\mu_{s,t}$ then up to a linear change of variables, the members of $\mathscr{P}$ are Jacobi polynomials with parameters α and β satisfying equation* (2).

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