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ACCURATE STRATEGIES FOR SMALL DIVISOR PROBLEMS

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I. Introduction

Many interesting problems in mechanics are close to systems that can be solved exactly. In such systems it is natural to consider perturbative expansions about the known system. Frequently it is possible to compute all of the terms in the expansion recursively, which can lead to expressions involving denominators that become arbitrarily small. The convergence of these expansions is difficult to establish and sometimes even false. In the 1960s, Kolmogorov, Arnold and Moser developed a systematic method, known as K.A.M. theory, to deal with these small divisor problems.

An unfortunate limitation of the K.A.M. theory is that the range of validity of the perturbation expansions that are established are very small in comparison to physically relevant values (see [Mo] in this regard). We have considered the problem of systematically improving the values yielded by K.A.M. theory. In two examples discussed below, we have proven lower bounds for the range of validity which are 90% of values for which the results are known to be false.

Two classical theorems whose proofs present all of the essential complications of K.A.M. are

Theorem I.1. *Given a family* f_ε *of analytic functions on* $\mathbf{C}$,

$$f_\varepsilon(z) = az + \frac{1}{\varepsilon}\hat{f}(\varepsilon z)$$

where $\hat{f}(z) = \mathscr{O}(z^2)$, *and, for some* C, $\nu > 0$,

$$|a| = 1, \qquad |a^n - 1|^{-1} \leq Cn^\nu \qquad \forall n > 0.$$

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