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SELF-MAPS OF CLASSIFYING SPACES OF COMPACT SIMPLE LIE GROUPS

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ABSTRACT. We describe here the set $[BG, BG]$ of homotopy classes of self-maps of the classifying space BG, for any compact connected simple Lie group G. In particular, we show that two maps $f, f' : BG \to BG$ are homotopic if and only if they are homotopic after restricting to the maximal torus of G; or equivalently if and only if they induce the same homomorphism in rational cohomology. In addition, we identify the homotopy types, up to profinite completion, of the components of the mapping space $\mathrm{map}(BG, BG)$.

The most central concern of homotopy theory is the classification, up to homotopy, of maps between topological spaces. It has long been suspected that maps between classifying spaces provide a particularly favorable special case of this problem, in which explicit results can be expected. In this paper we announce a complete classification of the self-maps of the classifying space BG, when G is any compact connected simple Lie group.

When G and Γ are arbitrary compact Lie groups, then $[BG, B\Gamma]$ will denote the set of unbased homotopy classes of maps from BG to $B\Gamma$. It is natural to ask how closely this set is related to the set $\mathrm{Hom}(G, \Gamma)$ of homomorphisms from G to Γ. For any inner automorphism $\alpha \in \mathrm{Inn}(\Gamma)$, $B\alpha$ is homotopic to the identity on $B\Gamma$. It is thus convenient to write $\mathrm{Rep}(G, \Gamma) = \mathrm{Hom}(G, \Gamma)/\mathrm{Inn}(\Gamma)$; and ask when the map

$$B : \mathrm{Rep}(G, \Gamma) \to [BG, B\Gamma]. \qquad B(\rho) = B\rho$$

is a bijection.

When G and Γ are both finite (or even discrete), then B is easily seen to be bijective. A much deeper result, due to Dwyer and Zabrodsky [2], says that B is bijective whenever G is a p-group, (and Γ is compact Lie). This was extended by Notbohm [8] to the case where G is a p-toral group: i.e., where G has toral identity

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