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The book of prime number records, by Paulo Ribenboim. Springer-Verlag, New York, Berlin, Heidelberg, 1988, xxiii + 476 pp., $49.80. ISBN 0-387-96573-4

Do people ever ask you, "What is the largest known prime?" or "For how many zeros of the zeta function is the Riemann Hypothesis known to hold?" Now at last there is a book which answers many questions of this type.

Guinness published the *Book of World Records* to settle arguments. Ribenboim has written a similar book to settle arguments about prime numbers. The world would be very civilized indeed if a brawl in a pub began with a dispute about how many twin primes are known. Some of the records which are reported are the results of computer searches and some of them are statements of theorems which approach best possible results.

Euclid's proof that there are infinitely many primes is well known to most mathematicians: If there were only finitely many primes, list all of them, multiply them together and add 1. The resulting number is not divisible by any prime on the list. However, either it is prime or it has a prime factor not on the list. In either case, there exists a prime not on the list which was supposed to contain all primes.

Some students misunderstand this proof and believe that it says that if you multiply together all the primes up to some point, and add 1, the result must be prime. Write $p\#$ for the product of all primes $\leq p$. Then $p\#+1$ is indeed prime for $p = 2, 3, 5, 7$ and 11, which fosters the students' misconception. However, $13\#+1$ is not prime. Is $p\#+1$ ever prime again? Yes, it is prime also for $p = 31$, 379, 1019, 1021, 2657, 3229, 4547, 4787, 11549 and 13649. The last five examples were discovered by Dubner [**4**] in 1987. The record largest known p for which $p\#+1$ is prime is $p = 13649$.

Write $\pi(x)$ for the number of primes up to x. A simple approximation for $\pi(x)$ is $\pi(x) \approx x/\log x$. A closer approximation is $\pi(x) \approx \mathrm{Li}(x) = \int_0^x dt/\log t$. What is the largest x for which $\pi(x)$ has been computed exactly? The record is $\pi(4 \times 10^{16}) = 1{,}075{,}292{,}778{,}753{,}150$, computed by Lagarias, Miller and Odlyzko [**6**] in 1985. They used a supercomputer, of course, but it did not generate each of these primes. They used an inclusion-exclusion technique based on a formula devised by Meissel in 1871 (and improved by several researchers since then). It is interesting to note that they chose the algorithm which is asymptotically *second* fastest. (The overhead of the algorithm which is ultimately fastest makes it slower than the second best one for numbers which are small enough to perform the calculation at all.)

The distribution of prime numbers is closely connected to the location of the zeros of the Riemann zeta function. This function of a complex variable has infinitely many zeros with real part between 0 and 1. If all